Exercise 6

Find the series solution for the following inhomogeneous second order ODEs:

$$u'' - xu' + xu = e^x$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Also, the Taylor series of e^x about $x = 0$ is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Now we substitute these series into the ODE.

$$u'' - xu' + xu = e^x$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$. In addition, the second series is zero for $n = 0$, so we can start the sum from $n = 1$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Since we want to combine the series, we want the first two series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$, and we can start the second at $n = 0$ as long as we replace n with $n + 1$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

To get x^{n+1} in the first series on the left, write out the first term and change n to $n + 1$. Do the same with the series on the right side of the equation.

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 1 + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{n+1}$$

Now that we have x^{n+1} in every series, we can combine the series.

$$2a_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3}x^{n+1} - (n+1)a_{n+1}x^{n+1} + a_n x^{n+1}] = 1 + \sum_{n=0}^{\infty}\frac{1}{(n+1)!}x^{n+1}$$

Factor the left side.

$$2a_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3} - (n+1)a_{n+1} + a_n]x^{n+1} = 1 + \sum_{n=0}^{\infty}\frac{1}{(n+1)!}x^{n+1}$$

Now we match coefficients on both sides.

$$\begin{aligned} &2a_2 = 1 \\ &(n+3)(n+2)a_{n+3} - (n+1)a_{n+1} + a_n = \frac{1}{(n+1)!} \end{aligned}$$

Now that we know the recurrence relations, we can determine a_n.

$$\begin{array}{rrcl} & 2a_2 = 1 & \rightarrow & a_2 = \frac{1}{2} \\ n=0: & 6a_3 - a_1 + a_0 = 1 & \rightarrow & a_3 = \frac{1}{6}(1 - a_0 + a_1) \\ n=1: & 12a_4 - 2a_2 + a_1 = \frac{1}{2} & \rightarrow & a_4 = \frac{1}{24}(3 - 2a_1) \\ n=2: & 20a_5 - 3a_3 + a_2 = \frac{1}{6} & \rightarrow & a_5 = \frac{1}{120}(1 - 3a_0 + 3a_1) \\ n=3: & 30a_6 - 4a_4 + a_3 = \frac{1}{24} & \rightarrow & a_6 = \frac{1}{720}(9 + 4a_0 - 12a_1) \\ n=4: & 42a_7 - 5a_5 + a_4 = \frac{1}{120} & \rightarrow & a_7 = \frac{1}{5040}(-9 - 15a_0 + 25a_1) \\ n=5: & 56a_8 - 6a_6 + a_5 = \frac{1}{720} & \rightarrow & a_8 = \frac{1}{40320}(49 + 42a_0 - 90a_1) \\ & & \vdots & \vdots \end{array}$$

Therefore,

$$\begin{aligned} u(x) = a_0 &\left(1 - \frac{1}{6}x^3 - \frac{1}{40}x^5 + \frac{1}{180}x^6 - \frac{1}{336}x^7 + \frac{1}{960}x^8 + \cdots\right) \\ &+ a_1\left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{40}x^5 - \frac{1}{60}x^6 + \frac{5}{1008}x^7 - \frac{1}{448}x^8 + \cdots\right) \\ &+ \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{8}x^4 + \frac{1}{120}x^5 + \frac{1}{80}x^6 - \frac{1}{560}x^7 + \frac{7}{5760}x^8 + \cdots, \end{aligned}$$

where a_0 and a_1 are arbitrary constants.